Physical-Measure Future Value Distributions

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Presentation Overview

- We distinguish here between $\mathbb{P}$ and the risk-neutral measure, $\widetilde{\mathbb{P}}$, a special measure constructed to facilitate risk-neutral pricing

$$\frac{V(t)}{\beta(t)} = \widetilde{\mathbb{E}}\left[\frac{V(T)}{\beta(T)}\middle|\mathcal{F}(t)\right],$$

  where $V(t)$ is value of claim, and $\beta(t)$ is money market account.
- The physical measure characterises the probabilities of various scenarios playing out in the *real world*.
- Banks and life companies now face the problem of generating future value distributions for their portfolio/s under the physical measure, $\mathbb{P}$.
- For the purposes of assessing future portfolio behaviour (risk *etc.*), the physical measure is often more relevant, *i.e.* we may seek quantities such as

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$$\text{or } \mathbb{P}\big(V^{(p)}(T) - V^{(h)}(T) > L\big) \quad p = \text{portfolio}, \; h = \text{hedge}.$$

Example of Physical and Risk-Neutral Exposures Requiring American Monte Carlo.

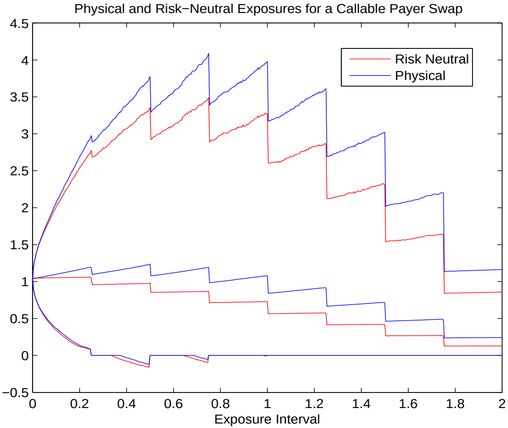

Figure: Physical and risk-neutral daily (5%, mean, 95%) exposures for a quarterly **callable** payer IRS. Notional 100, tenor 2Y. HW1F('91) model parameters $r(t_0) = 0.05$, $\kappa = 0.5$, $r_\infty = 0.05$, $\tilde{r}_\infty = 0.04$, $\sigma = 0.01$.

Agenda

1. Exploring exposures under alternative measures with a few simple examples.
2. Change-of-measure processes and pricing kernels.
3. American Monte Carlo-based risk-neutral exposures.
4. American Monte Carlo-based physical exposures using a change-of-measure process.
5. PFE, CVA and funding cost calculations for a vanilla IRS.
6. Estimation of risk-premia parameters.

Introductory Example within a Black Scholes (‘73) Economy. 1.

- Consider an equity index, $S(t)$, evolving under $\widetilde{\mathbb{P}}$ according to the Black Scholes (‘73) model,
$$\frac{dS}{S} = r\,dt + \sigma\,d\widetilde{B}\,,$$
where $r = r(t)$, $\sigma = \sigma(t)$, and $\widetilde{B} = \widetilde{B}(t)$ is a $\widetilde{\mathbb{P}}$-Brownian Motion.
- Let $C(S,t)$ be the pricing function for a call option. By the Fundamental Pricing PDE we know that
$$rC = C_t + C_S r S + \frac{1}{2}C_{SS}\sigma^2 S^2\,, \text{ or } r = \frac{C_t + C_S r S + \frac{1}{2}C_{SS}\sigma^2 S^2}{C}\,,$$
and by Ito's Lemma we know that the *return* on the call is
$$\frac{dC}{C} = \left(\frac{C_t + C_S r S + \frac{1}{2}C_{SS}\sigma^2 S^2}{C}\right)dt + \left(\frac{C_S \sigma S}{C}\right)d\widetilde{B}\,.$$

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$$\frac{dC}{C} = \Bigg(\underbrace{\frac{C_t + C_S rS + \frac{1}{2}C_{SS}\sigma^2 S^2}{C}}_{=r}\Bigg)dt + \left(\frac{C_S\sigma S}{C}\right)d\widetilde{B} \implies \widetilde{\mathbb{E}}\left[\frac{dC}{C}\,\middle|\,\mathcal{F}(t)\right] = r\,dt\,.$$

Introductory Example within a Black Scholes ('73) Economy. 2.

- Consider the evolution of $S(t)$ under $\mathbb{P}$, with $B = B(t)$ a $\mathbb{P}$-Brownian Motion,

$$\frac{dS}{S} = \mu\, dt + \sigma\, dB\,.$$

- Here, $\mu = \mu(t)$, and let $\theta = \theta(t)$ denote the price per unit of B risk,

$$\theta = \frac{\mu - r}{\sigma}\,, \text{ or } \mu = r + \sigma\theta\,.$$

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Introductory Example within a Black Scholes ('73) Economy. 3.

- Natural to assume that $\theta > 0$, *i.e.* a positive equity risk premium.
- Importantly, being a call option we have $\dfrac{\partial C}{\partial S} > 0$. Thus,

$$\mathbb{E}\left[\frac{dC}{C}\middle|\mathcal{F}(t)\right] = \left(r + \underbrace{\left(\frac{C_S \sigma S}{C}\right)\theta}_{(+)\times(+)=(+)} \right) dt > r\, dt\,.$$

- If we were working with a put option, $P(S,t)$, then $\dfrac{\partial P}{\partial S} < 0$. Thus,

$$\mathbb{E}\left[\frac{dP}{P}\middle|\mathcal{F}(t)\right] = \left(r + \underbrace{\left(\frac{P_S \sigma S}{P}\right)\theta}_{(-)\times(+)=(-)} \right) dt < r\, dt\,.$$

- In reality, call buyers earn a premium and put payers pay a premium (insurance), causing the $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ future value distributions to divege.

Introductory Example within a Black Scholes ('73) Economy. 3.

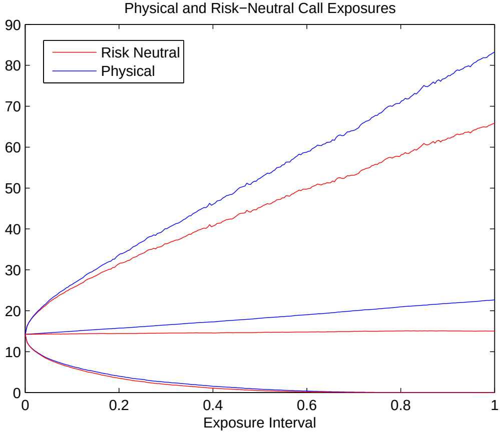

Figure: Physical and risk-neutral (5%, mean, 95%) daily exposures for a European call contract. Initial stock 100, strike 100, tenor 1Y. Model parameters $r = 0.05$, $\sigma = 0.3$, $\phi = 0.1$.

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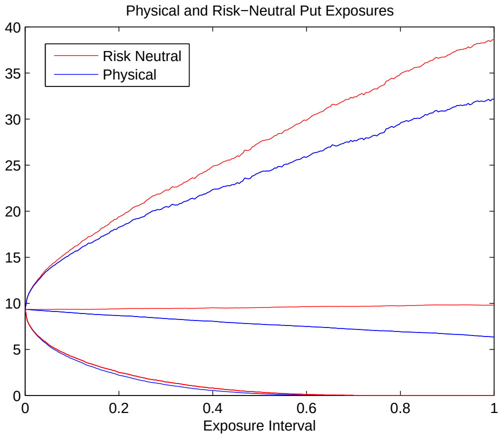

Figure: Physical and risk-neutral (5%, mean, 95%) daily exposures for a European put contract. Initial stock 100, strike 100, tenor 1Y. Model parameters $r = 0.05$, $\sigma = 0.3$, $\phi = 0.1$.

Introductory Example within a Black Scholes ('73) Economy. 5.

- The divergence can be more extreme if *leverage* is considered.
- Form a self-financing portfolio leveraged portfolio in the call, with value $V^{(p)}(t)$, initial leverage ratio $L(t_0)$, and initial cost $V^{(p)}(t_0) = C(t_0)(1 - L(t_0))$.
- Leverage ratio at arbitrary time t is $L(t) = \dfrac{C(t) - V^{(p)}(t)}{C(t)}$.
- Portfolio allocation is 1 unit of $C(t)$, and $\dfrac{V^{(p)} - C(t)}{\beta(t)}$ units of $\beta(t)$,

$$\frac{dV^{(p)}}{V^{(p)}} = r\,dt + \left(\frac{1}{(1-L)}\left(\frac{C_S \sigma S}{C}\right)\right)\theta\,dt + \left(\frac{1}{(1-L)}\left(\frac{C_S \sigma S}{C}\right)\right)dB\,.$$

- Depending on $L(t_0)$, $\mathbb{E}\left[\frac{dV^{(p)}}{V^{(p)}}\middle|\mathcal{F}(t)\right]$ can be *much* larger than $r\,dt$.

Introductory Example within a Black Scholes ('73) Economy. 6.

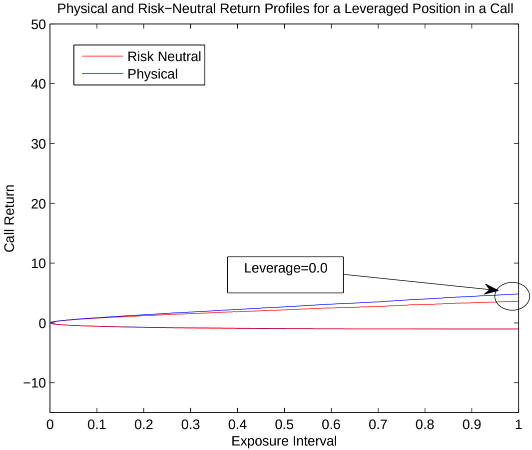

Figure: Physical and risk-neutral (5%, mean, 95%) daily exposures for a leveraged portfolio in the European put contract. Leverage ratio $L = 0.0$.

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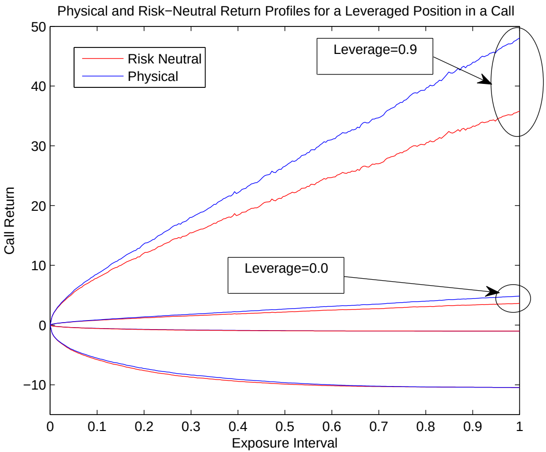

Figure: Physical and risk-neutral (5%, mean, 95%) daily exposures for a leveraged portfolio in the European put contract. Leverage ratios $L = 0.0$ and $L = 0.9$.

Introductory Example within a Heston ('93) Economy. 1.

- Introduce a stochastic volatility process, $H(t)$, with $S(t)$ evolving under $\mathbb{P}$ as

$$\begin{aligned} \frac{dS}{S} &= (r + \phi H)\,dt + \sqrt{H}\,dB_1 \\ dH &= \kappa(H_\infty - H)\,dt + \xi\sqrt{H}(\rho\,dB_1 + \sqrt{1-\rho^2}\,dB_2)\,. \end{aligned}$$

- By Ito's Lemma we have the returns process for a call, $C(S, H, t)$,

$$\begin{aligned} \frac{dC}{C} &= \cdots dt + \left(\frac{C_S\sqrt{H}}{C}\right) dS + \left(\frac{C_H\xi\sqrt{H}}{C}\right) dH \\ &= \cdots dt + \left(\frac{C_S\sqrt{H}S + C_H\xi\sqrt{H}\rho}{C}\right) dB_1 + \left(\frac{C_H\xi\sqrt{H}\sqrt{1-\rho^2}}{C}\right) dB_2\,. \end{aligned}$$

- Provided the option is spanned by the market, and assuming no arbitrage,

$$\begin{aligned} \mathbb{E}\left[\frac{dC}{C}\middle|\mathcal{F}(t)\right] &= r\,dt + \left(\frac{C_S\sqrt{H}S}{C}\right)\Phi\,dt + \left(\frac{C_H\xi\sqrt{H}}{C}\right)\Gamma\,dt \\ &= r\,dt + \left(\frac{C_S\sqrt{H}S + C_H\xi\sqrt{H}\rho}{C}\right)\theta_1\,dt + \left(\frac{C_H\xi\sqrt{H}\sqrt{1-\rho^2}}{C}\right)\theta_2\,dt\,. \end{aligned}$$

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Introductory Example within a Heston ('93) Economy. 2.

- Here $\{\Phi, \Gamma\}$ are premia for $\{S, H\}$ risk, $\{\theta_1, \theta_2\}$ are premia for $\{B_1, B_2\}$ risk.
- Assume as in Heston ('93) that the *volatility premium* $\Gamma = \gamma H$, $\gamma \leq 0$.
- The corresponding $\widetilde{\mathbb{P}}$ under which all traded portfolios earn the short rate is

$$\begin{aligned} \frac{dS}{S} &= r\,dt + \sqrt{H}\,d\widetilde{B}_1 \\ dH &= \underbrace{\widetilde{\kappa}(\widetilde{H}_\infty - H)}_{\neq r}\,dt + \xi\sqrt{H}(\rho\,d\widetilde{B}_1 + \sqrt{1-\rho^2}\,d\widetilde{B}_2)\,, \end{aligned}$$

where

$$\widetilde{\kappa} = \kappa + \gamma\,, \quad \text{and } \widetilde{H}_\infty = H_\infty \frac{\kappa}{\kappa+\gamma}\,.$$

- Now *both* $\dfrac{\partial C}{\partial H} > 0$ and $\dfrac{\partial P}{\partial H} > 0$, so *both* instruments command premium (<0).
- Set up self-financing Δ-hedged portfolio in call, $V^{(h)}$, *i.e.* $\dfrac{\partial V^{(h)}}{\partial S} = 0$. Thus *only* volatility risk remains.

Introductory Example within a Heston ('93) Economy. 3.

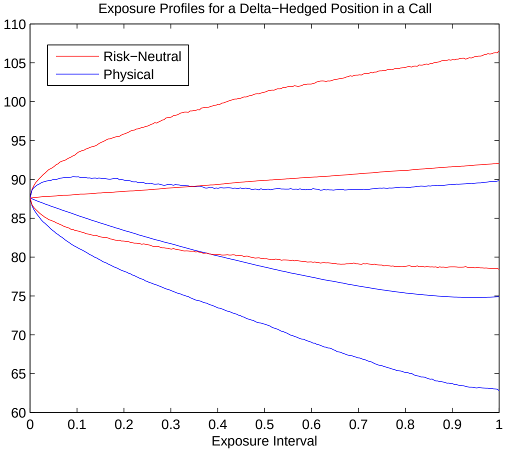

Figure: Physical and risk-neutral (5%, mean,95%) daily exposures for a self-financing delta-hedged portfolio in the European call contract in a Heston ('93) model. Initial stock 500, strike 500. Premia parameters $\phi = 0.25$, $\gamma = -1.5$.

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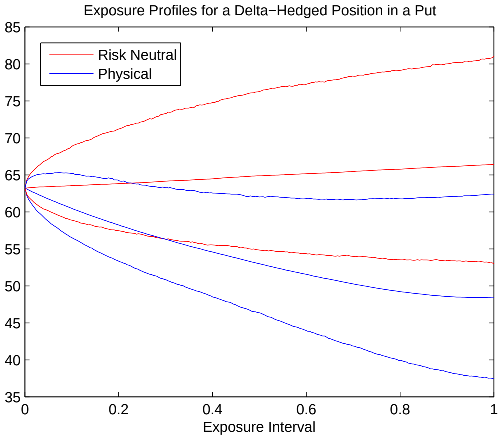

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The Change-of-Measure Process and the Pricing Kernel. 1.

- For general case, let $X(t) \in \mathbb{R}^D$ denote the state vector, evolving under $\mathbb{P}$ as

$$dX = \mu_X(X,t)\,dt + \Sigma_X^{1/2}(X,t)\,dB\,.$$

- Let $Z(t)$ be the $\mathbb{P}$-into-$\widetilde{\mathbb{P}}$ Radon Nikodym derivative process, with the property $\dfrac{Z(T)}{Z(t)} = \dfrac{d\widetilde{\mathbb{P}}(\,\cdot\,|\mathcal{F}(t))}{d\mathbb{P}(\,\cdot\,|\mathcal{F}(t))}$ for $T \geq t$, evolving under $\mathbb{P}$ as

$$\frac{dZ}{Z} = -\theta^{\mathrm{T}}(X,t)\,dB\,,$$

  where $\theta(t) \in \mathbb{R}^D$ is a vector of risk premia.

- Given technical conditions, Girsanov's Theorem states that the $\widetilde{\mathbb{P}}$ evolution of $X(t)$ is

$$dX = \big(\mu_X(X,t) - \Sigma_X^{1/2}(X,t)\theta(X,t)\big)\,dt + \Sigma_X^{1/2}(X,t)\,d\widetilde{B}\,.$$

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The Change-of-Measure Process and the Pricing Kernel. 2.

- By definition, the risk-neutral pricing formula can be recast as

$$\frac{V(t)}{\beta(t)} = \widetilde{\mathbb{E}}\left[\frac{V(T)}{\beta(T)}\middle|\mathcal{F}(t)\right] = \mathbb{E}\left[\frac{V(T)}{\beta(T)}\frac{Z(T)}{Z(t)}\middle|\mathcal{F}(t)\right].$$

- This motivates the definition of a pricing kernel, $\Lambda(t) = \dfrac{Z(t)}{\beta(t)}$,

$$\frac{V(t)}{\Lambda^{-1}(t)} = \mathbb{E}\left[\frac{V(T)}{\Lambda^{-1}(T)}\middle|\mathcal{F}(t)\right], \text{ or } \mathbb{E}\left[d\left(\frac{V(t)}{\Lambda^{-1}(t)}\right)\middle|\mathcal{F}(t)\right] = 0\,.$$

- The process Λ^{-1} is essentially a numeráire for the *physical measure*.
- The pricing kernel satisfies

$$\frac{d\Lambda}{\Lambda} = -r\,dt - \theta^{\mathrm{T}}\,dB\,.$$

- Where do the prices of risk within θ actually come from? Functional form?
- Martingale pricing theory guarantees the existence of $\widetilde{\mathbb{P}}$ in a no-arbitrage environment. No need to specify θ, just $\widetilde{\mu}_X = \mu_X - \Sigma_X^{1/2}\theta$.

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numerix

The Monte Carlo-on-Monte Carlo Approach. 1.

- Require a $\mathbb{P}$- (or $\widetilde{\mathbb{P}}$-) measure future values "cube", $\{V_n(t_i)\}_{i=0}^{T}$, $n = 1, \cdots, N$.
- Allows for approximation of loss percentiles, expected exposures *etc.* via

$$\mathbb{P}(V(t_i) < L|\mathcal{F}(t_0)) \approx \frac{1}{N}\sum_{n=1}^{N} 1_{(V_n(t_i)<L)} \text{ or } \widetilde{\mathbb{E}}\left[(V(t_i))^+ \middle| \mathcal{F}(t_0)\right] \approx \frac{1}{N}\sum_{n=1}^{N}(V_n(t_i))^+ .$$

- Can simulate $\{X_n(t_i)\}_{i=0}^{T}$, $n = 1, \cdots, N$, using $\mathbb{P}$- (or $\widetilde{\mathbb{P}}$-) measure dynamics, and value the deal on each "node", $X_n(t_i)$.
- For tractable models and vanilla trades, there may exist closed-form functions.
- For more complicated models or structured trades, future values must be produced by Monte Carlo *on each node*. Computationally burdensome.
- Have $(N \times T)$ MC runs, each with N paths and (approx.) $\dfrac{T}{2}$ timesteps, essentially $\mathcal{O}(N^2)$.[1]

[1] T can be large too for long-dated portfolios and frequent margining.

Risk-Neutral Future Values by American Monte Carlo. 1.

- American Monte Carlo, based upon principle of Longstaff Schwartz (2001) regression, commonly used to value callable products.
- For intrinsic value function $\Psi(X(t_i))$, value satisfies recursions for $i \leq T-1$,

$$\frac{V(t_i)}{\beta(t_i)} = \max\left(\frac{\Psi(X(t_i))}{\beta(t_i)}, \underbrace{\widetilde{\mathbb{E}}\left[\frac{V(t_{i+1})}{\beta(t_{i+1})}\middle|\mathcal{F}(t_i)\right]}_{\text{continuation value}}\right), \text{ with } V(t_T) = \Psi(X(t_T)).$$

- AMC "rolls back" deflated continuation values, $\dfrac{V^{(c)}(t_i)}{\beta(t_i)}$, using Monte Carlo simulation and linear projection.
- The continuation values are projected onto a set of basis function $\{f_m(\cdot)\}_{m=0}^{M}$,

$$\frac{V^{(c)}(t_i)}{\beta(t_i)} = \widetilde{\mathbb{E}}\left[\frac{V(t_{i+1})}{\beta(t_{i+1})}\middle|\mathcal{F}(t_i)\right] \approx \sum_{m=0}^{M} c_m f_m(X(t_i)).$$

Risk-Neutral Future Values by American Monte Carlo. 1.

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- For intrinsic value function $\Psi(X(t_i))$, value satisfies recursions for $i \leq T-1$,

$$\frac{V(t_i)}{\beta(t_i)} = \max\left(\frac{\Psi(X(t_i))}{\beta(t_i)}, \underbrace{\widetilde{\mathbb{E}}\left[\frac{C(t_{i+1})}{\beta(t_{i+1})} \middle| \mathcal{F}(t_i) \right]}_{\text{committed flow}} + \underbrace{\widetilde{\mathbb{E}}\left[\frac{V(t_{i+1})}{\beta(t_{i+1})} \middle| \mathcal{F}(t_i) \right]}_{\text{continuation value}} \right).$$

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$$\frac{V^{(c)}(t_i)}{\beta(t_i)} = \widetilde{\mathbb{E}}\left[\frac{V(t_{i+1})}{\beta(t_{i+1})}\middle|\mathcal{F}(t_i)\right] = \underbrace{\widetilde{\mathbb{E}}\left[\frac{V(t_{i+1})}{\beta(t_{i+1})}\middle|X(t_i)\right]}_{\text{assumes path independence}} \approx \sum_{m=0}^{M} c_m f_m(X(t_i)).$$

Risk-Neutral Future Values by American Monte Carlo. 2.

- Simulate N paths of the state process, $\{X_n(t_i)\}_{i=0}^{T}$, $n = 1, \cdots N$, *under* $\widetilde{\mathbb{P}}$.
- Terminal values known at t_T, $\hat{V}_n(t_T) = \hat{V}_n^{(c)}(t_T) = \Psi(X_n(t_T))$, $n = 1, \cdots, N$.
- We want to write
$$\frac{\hat{V}_n^{(c)}(t_i)}{\beta_n(t_i)} = \sum_{m=0}^{M} \hat{c}_m f_m(X_n(t_i)).$$
- The estimated coefficients $\{\hat{c}_m\}_{m=0}^{M}$ are recovered by regression, minimising $\sum_n e_n^2(t_i)$ in
$$\frac{\hat{V}_n(t_{i+1})}{\beta_n(t_{i+1})} = \sum_{m=0}^{M} c_m f_m(X_n(t_i)) + e_n(t_i)\,.$$
- Can use the same technique to "roll back" values in general. Produces all values on a "time slice" in one regression, *i.e.* $\{\hat{V}_n(t_i)\}_{n=1}^{N}$. A $\widetilde{\mathbb{P}}$-measure cube.
- Essentially, have T-many regressions, with cost depending on N and M, typically $\mathcal{O}(NM^2)$. See Cesari *et al.* ('07) or Antonov *et al.* ('11).

Physical Future Values by American Monte Carlo. 1.

- The goal is to produce a $\mathbb{P}$-measure cube, so work with a $\mathbb{P}$ expectation,

$$\frac{V(t_i)Z(t_i)}{\beta(t_i)} = \mathbb{E}\left[\frac{V(t_{i+1})Z(t_{i+1})}{\beta(t_{i+1})}\middle|\mathcal{F}(t_i)\right] \approx \sum_{m=0}^{M} k_m f_m(X(t_i)) \,.$$

- Simulate N paths of $\{X_n(t_i)\}_{i=0}^{T}$, *and* $\{Z_n(t_i)\}_{i=0}^{T}$ *or* $\{\Lambda_n(t_i)\}_{i=0}^{T}$, *all under* $\mathbb{P}$.
- Using the $\mathbb{P}$ expectation write (could also be likened to importance sampling!)

$$\frac{\hat{V}_n(t_i)Z_n(t_i)}{\beta_n(t_i)} \approx \sum_{m=0}^{M} \hat{k}_m f_m(X_n(t_i)) \,,$$

  Antonov *et al.* ('11) introduce $Z(t)$ via a fictitious FX process at this step.
- Estimate $\{\hat{k}_m\}_{m=0}^{M}$ by minimising $\sum_{n=1}^{N} u_n^2$ in

$$\frac{\hat{V}_n(t_{i+1})Z_n(t_{i+1})}{\beta_n(t_{i+1})} = \sum_{m=0}^{M} k_m f_m(X_n(t_i)) + u_n(t_i) \,.$$

- The set $\{\hat{V}_n(t_i)\}_{n=1}^{N}$ approximates a future values "time slice" under $\mathbb{P}$, with essentially the same computational burden as in the $\widetilde{\mathbb{P}}$ case.

Physical Future Values by American Monte Carlo. 2.

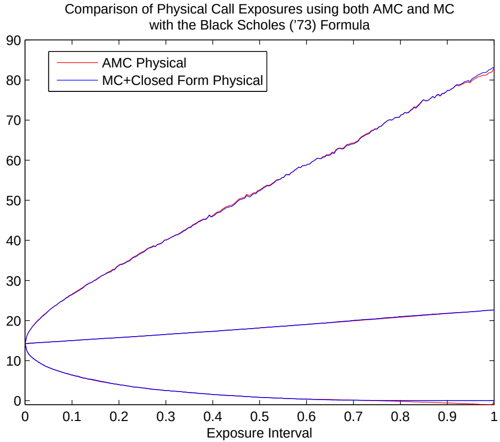

Figure: Physical daily exposures for a European call contract. Initial stock 500, strike 500, tenor 1Y. Model parameters $r = 0.05$, $\sigma = 0.3$. 20,000 paths (N), 5 monomials (M).

Physical Future Values by American Monte Carlo. 3.

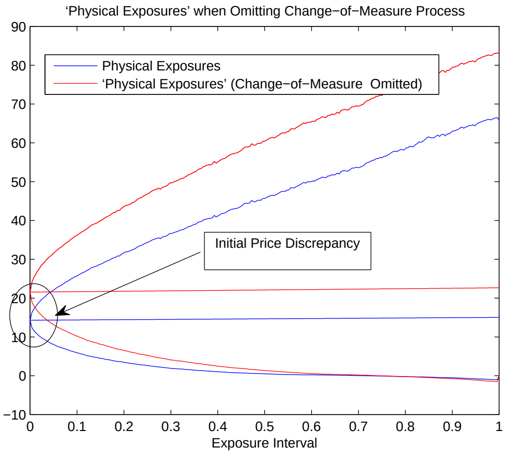

Figure: Physical-measure exposures when the change-of-measure process is omitted during the AMC algorithm.

Physical Future Values by American Monte Carlo. 4.

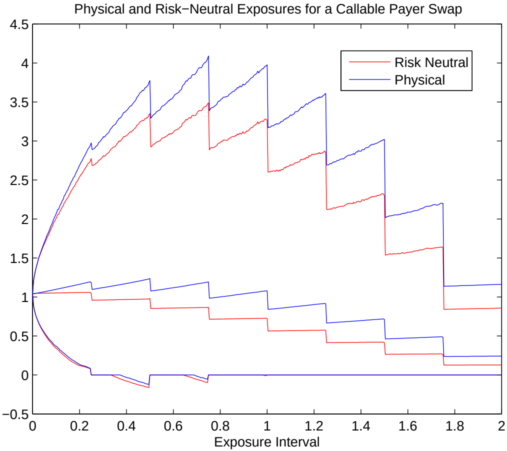

Figure: Physical and risk-neutral daily (5%, mean, 95%) exposures for a quarterly **callable** payer IRS. Notional 100, tenor 2Y. Model parameters $r(t_0) = 0.05$, $\kappa = 0.5$, $r_\infty = 0.05$, $\widetilde{r}_\infty = 0.04$, $\sigma = 0.01$.

Physical Future Values by American Monte Carlo. 5.

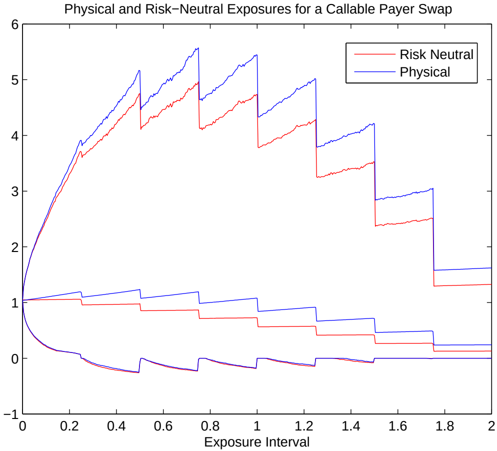

Figure: Physical and risk-neutral daily (1%, mean, 99%) exposures for a quarterly **callable** payer IRS. Notional 100, tenor 2Y. Model parameters $r(t_0) = 0.05$, $\kappa = 0.5$, $r_\infty = 0.05$, $\tilde{r}_\infty = 0.04$, $\sigma = 0.01$.

Physical Future Values by American Monte Carlo. 6.

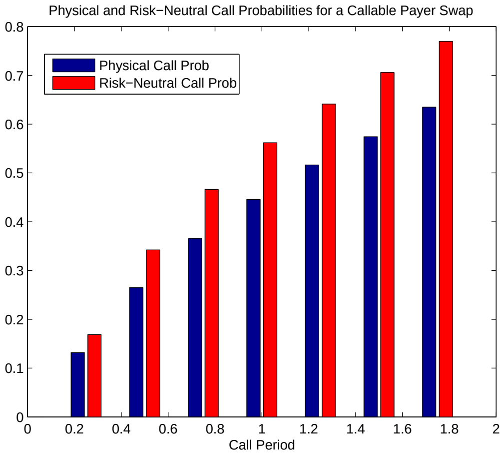

Figure: Physical and risk-neutral call probabilities for a quarterly **callable** payer IRS. Notional 100, tenor 2Y. Model parameters $r(t_0) = 0.05$, $\kappa = 0.5$, $r_\infty = 0.05$, $\tilde{r}_\infty = 0.04$, $\sigma = 0.01$.

PFE and CVA Computations. 1.

- CVA and PFE figures require knowledge of counterparty "exposure" behaviour.
- Let $V^{(p)}(t)$ be portfolio value with, $C(t)$ the collateral account.
- Collateral posted at t_i depends on $C(t_{i-1})$ and thresholds, MTAs, rounding *etc*.
- Define exposure on interval (t_i, t_{i+1}) as $E(t_i, t_{i+1}) = V^{(p)}(t_{i+1}) - C(t_{i+1})\ (^{+})$.
- Let $\tau^{(s)}$ be self default time, $\tau^{(c)}$ counterparty's, $\lambda^{(s)}$ and $\lambda^{(c)}$ hazard rates.
- CVA is defined as a type of $\widetilde{\mathbb{P}}$-expected discounted loss given default,

$$\mathrm{CVA}(t_0) = \widetilde{\mathbb{E}}\left[\sum_{i=0}^{T} 1(\tau^{(s)} > t_i) 1(t_{i+1} \geq \tau^{(c)} > t_i) \frac{\beta(t_0)}{\beta(t_{i+1})} E(t_i, t_{i+1}) \middle| \mathcal{F}(t)\right].$$

- The time-t_i 95% PFE is the 95^{th} percentile of the distribution of $E(t_{i-1}, t_i)$.
- Limits on maximum PFEs are imposed to manage counterparty credit risk.
- Ideally, PFEs are to be computed as percentiles of the $\mathbb{P}$-measure distribution.

PFE and CVA Computations. 2.

- CVA approximations are of the type (ignore self-survival effect for simplicity)

$$\mathrm{CVA}(t_0) \approx \frac{1}{N}\sum_{n=1}^{N}\left(\sum_{i=0}^{T} e^{-\sum_{j=0}^{i-1}\lambda_n^{(c)}(t_j)\Delta_j}\left(1-e^{-\lambda_n(t_i)\Delta_i}\right)\frac{\beta_n(t_0)}{\beta_n(t_{i+1})}\left(V_n^{(p)}(t_{i+1})-C_n(t_{i+1})\right)\right),$$

which requires $\widetilde{\mathbb{P}}$-measure slices $\{V_n^{(p)}(t_i)\}_{n=1}^{N}$ as pre-computed inputs.

- Similarly, PFE-related probabilities require approximations of the type

$$\mathrm{PFE}(\alpha\%, t_i) = L : \ \frac{1}{N}\sum_{n=1}^{N} 1\left(V_n^{(p)}(t_i) > L\right) = \alpha\%\,.$$

which requires $\mathbb{P}$-measure slices $\{V_n^{(p)}(t_i)\}_{n=1}^{N}$ as pre-computed inputs.

Example using a Vanilla IRS and a HW1F ('90) Model. 1.

- Let $V(t_i)$ be value of a payer swap with cashflows $(L(t_i) - R)\,\delta_j,\ i = 1, \cdots, T$.
- Take a HW1F ('90) model of the short rate $r(t)$, with $\mathbb{P}$ dynamics

$$dr = \kappa(r_\infty - r)\,dt + \sigma\,dB\,,$$

with $\kappa = \kappa(t)$, $r_\infty = r_\infty(t)$, and $\sigma = \sigma(t)$.

- Introducing $\theta = \theta(t)$ results in a $\widetilde{\mathbb{P}}$ system

$$dr = \kappa(\widetilde{r}_\infty - r)\,dt + \sigma\,d\widetilde{B}\,,$$

with

$$\widetilde{r}_\infty(t) = r_\infty(t) - \frac{\sigma(t)\theta(t)}{\kappa(t)}\,.$$

- The risk-neutral measure $\widetilde{\mathbb{P}}$ **overweights undesirable states**, when the short rate tends to be driven lower by demand for safe assets, hence choose θ such that $\widetilde{r}_\infty < r_\infty$, *i.e.* $\theta > 0$.

Example using a Vanilla IRS and a HW1F ('90) Model. 2.

- Choose θ to match the (time-dependent) difference in expected short rates, satisfying

$$\frac{1}{ds}\left(\mathbb{E}[r(s)|\mathcal{F}(t)] - \widetilde{\mathbb{E}}[r(s)|\mathcal{F}(t)]\right) = \sigma(s)\theta(s) + \kappa(s)\left(\mathbb{E}[r(s)|\mathcal{F}(t)] - \widetilde{\mathbb{E}}[r(s)|\mathcal{F}(t)]\right).$$

or

$$\mathbb{E}[r(s)|\mathcal{F}(t)] - \widetilde{\mathbb{E}}[r(s)|\mathcal{F}(t)] = \int_t^s \exp\left(\int_u^s \kappa(v)\,dv\right)\sigma(u)\theta(u)\,du\,.$$

- For piecewise $\kappa(t_i, t_{i+1})$, $\sigma(t_i, t_{i+1})$, the $\theta(t_i, t_{i+1})$ can be solved for recursively.
- Use a discretisation to simulate N joint series $(r_n(t_i), Z_n(t_i))$, $i = 1, ..., T$.[2]

$$\begin{aligned}\Delta r_n(t_i) &= \kappa(r_\infty(t_i) - r_n(t_i))\,\Delta_i + \sigma(t_i)\sqrt{\Delta_i}\,\epsilon_n(t_i)\\ \Delta Z_n(t_i) &= -Z_n(t_i)\theta(t_i, t_{i+1})\sqrt{\Delta_i}\,\epsilon_n(t_i)\,,\ \epsilon_n(t_i) \sim \text{iid N}(0,1)\,.\end{aligned}$$

[2] Here an Euler Maruyama discretisation is used, but one could sample from the joint distribution of $(r(t_{i+1}), Z(t_{i+1}))|(r(t_i), Z(t_i))$ directly in this case.

PFE and CVA Computations. 3.

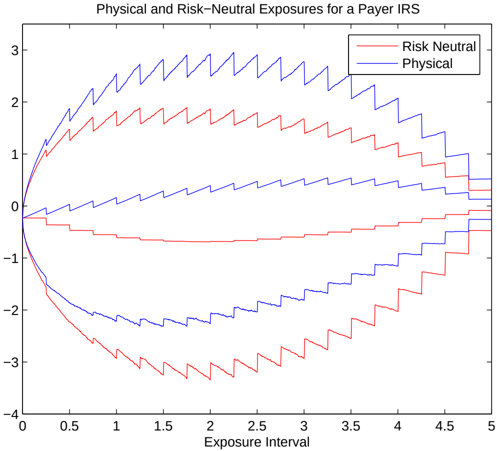

Figure: Physical daily (5%, mean, 95%) exposures for a payer IRS. Notional 100, tenor 5Y. Model parameters $r(t_0) = 0.05$, $\kappa = 0.5$, $r_\infty = 0.05$, $\tilde{r}_\infty = 0.04$, $\sigma = 0.01$.

PFE and CVA Computations. 4.

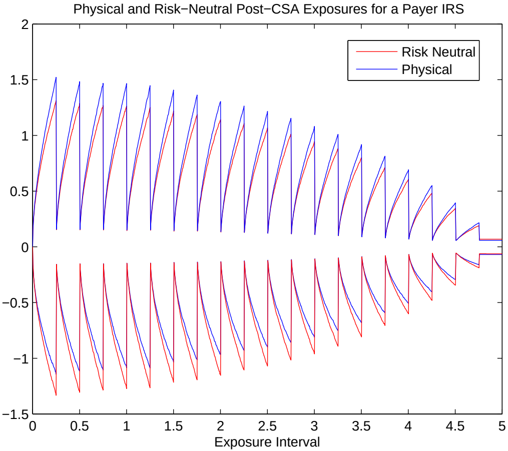

Figure: Physical daily (5%, 95%) post-CSA exposures for a payer IRS. Monthly margining, threshold 0, minimum transfer amount 0.1.

PFE and CVA Computations. 5.

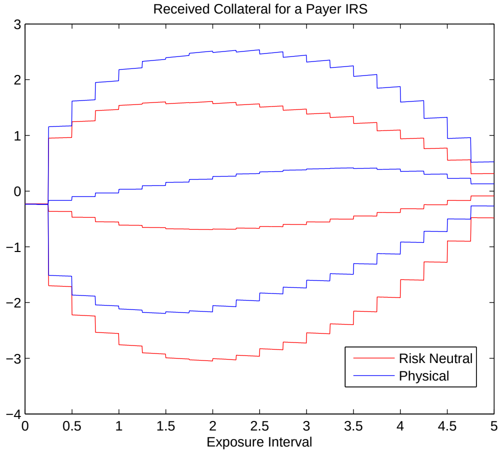

Figure: Physical daily (5%, mean, 95%) received collateral for a payer IRS. Monthly margining, threshold 0, minimum transfer amount 0.1.

Concluding Remarks

- Have seen impact of switching measure on risk calculations.
- Impact depends upon instrument sensitivities/factor premia/leverage *etc.*
- Can introduce a stochastic discount factor and value under the physical measure.
- Introduction of a stochastic discount factor allows for application of American Monte Carlo for production of future value distributions.
- Can have flow-through effects for value-at-risk figures, potential future exposures, expected funding costs, *etc.*

Estimation of Risk-Premia Parameters. 1.

- Can distinguish between the total parameter vector Θ and the risk-neutral *combinations* $\widetilde{\Theta}$ (*i.e.* $\widetilde{\kappa} = \kappa + \gamma$).
- Option prices are a function of $\widetilde{\Theta}$, *i.e.* $V(X, t; \widetilde{\Theta})$, but the dynamics of $X(t)$ are dependent upon Θ.
- In practice, we have a series of options data (or rates) and *some* factors (*i.e.* index levels and FX).
- Thus have information on the *full* Θ.
- Consider using maximum likelihood to estimate Θ.
- If we have the transitional PDF $f(X(t_{i+1})|X(t_i); \Theta)$, then the conditional PDF of the observed $V^{(o)}(t_{i+1})$ (with $\dim(V) = \dim(X)$) is

$$f(V^{(o)}(t_{i+1})|X(t_i); \Theta) = f(X^{-1}(V^{(o)}(t_{i+1}); \widetilde{\Theta})|X(t_i); \Theta) \left| \frac{\partial V(X; \widetilde{\Theta})}{\partial X} \right|^{-1}_{X^{-1}(V^{(o)}(t_{i+1}); \widetilde{\Theta})} .$$

- A function of both both physical and risk-neutral parameters.

Estimation of Risk-Premia Parameters. 2.

- Latent factors $\implies$ use *either* proxies *or* nonlinear filters and MCMC methods.

Proxies:

- Chernov and Ghysels ('00) use implied vol.'s for a Heston ('93) model via GMM/EMM
- Pan ('02) uses implied vol.'s for an extended Bates ('96) model via GMM.
- Ait-Sahalia and Kimmel ('07)/('08) use ML and VIX/yields for Heston ('93) and MF term-structure models.
- Santa Clara and Yan ('10) use implied vol.'s for an LQ stochastic intensity model via QML.

Filtering and MCMC Methods: (see review of Johannes and Polson ('06))

- Jones ('03) uses implied vol.'s for a CEV volatility model via MCMC .
- Eraker ('04)/Forbes, Martin and Wright ('07) uses SPX/single name options for Duffie *et al.* ('00)/Heston ('93) models via MCMC.
- Johannes, Polson and Stroud ('09) estimate a SVJJ model of Duffie *et al.* ('00) using a particle filter and SPX options.

The Change-of-Measure Process and the Pricing Kernel Cont'd. 3.

- As in Heston ('93) or Bates ('96), can obtain guidance from an equilibrium analysis.[3] Let $J(X,t)$ be the maximised indirect utility function,

$$J(W,X,t) = \max_{w} \mathbb{E}\left[U(W(T))\middle|\mathcal{F}(t)\right],$$

subject to wealth $W(t)$ evolving as a self-financing portfolio of assets $V(t) \in \mathbb{R}^M$,

$$dW = w \cdot dV \ \left(= \mu_W(W,X,w,t)\,dt + \sigma_W^{\mathrm{T}}(W,X,w,t)\,dB\right),$$

where $U(\cdot)$ is a utility function, $w = w(t) \in \mathbb{R}^M$ a control vector of holdings.

- $J(W,X,t)$ satisfies HJB equation

$$0 = \max_{w} \left\{ J_t + J_W\mu_W + J_X\mu_X + \frac{1}{2}J_{WW}\sigma_W^{\mathrm{T}}\sigma_W + \frac{1}{2}\mathrm{tr}[J_{XX}\Sigma_X] + J_{WX}\Sigma_X^{1/2}\sigma_W \right\}.$$

- First-order conditions with market clearing w imply that, for each $m = 1, \cdot, M$,

$$J_W V_m \mu_{V_m} - V_m J_W r + \left(J_{WW}\sigma_W^{\mathrm{T}} + J_{WX}^{\mathrm{T}}\Sigma_X^{1/2}\right)\sigma_V = 0 \implies \mathbb{E}[d(J_W V_m)] = 0\,.$$

[3] See for instance Merton ('73), Cox, Ingersoll and Ross ('85), Bates ('88) or Liu, Longstaff and Pan ('03), among others.

The Change-of-Measure Process and the Pricing Kernel Cont'd. 4.

- Note, the Heston ('93) volatility premium $\Gamma = \gamma H$ follows from using a power utility function over terminal wealth,

$$U(W(T)) = \frac{W^{1-\alpha}(T)}{1-\alpha} \implies J(W,H,t) = e^{a(\alpha,t)+b(\alpha,t)V(t)}W^{1-\alpha}(t)\,,$$

and

$$-\xi\rho H\frac{J_{WW}W}{J_W} - \xi^2 H\frac{J_{WH}}{J_W} = \underbrace{-\big(\xi b(\alpha,t)-\rho\alpha\big)\xi}_{=\gamma<0}\times H \propto H\,.$$

- Linear risk premia is no coincidence. Affine (or LQ) process with exponential-affine terminal conditions, *i.e.*

$$\mathbb{E}\big[U(W(T))\big|\mathcal{F}(t)\big] = \mathbb{E}\big[e^{-\log(1-\alpha)+(1-\alpha)\log(W(T))}\big|\mathcal{F}(t)\big]\,.$$

- Allows for the affine machinery of Duffie, Pan and Singleton ('00) or the LQ machinery of Leippold and Wu ('02) to be employed.
- Typically results in tractable $\widetilde{\mathbb{P}}$ systems.

An Alternative Approach

- Ostensibly, it is possible to work with only a single cube, and a change-of-measure process.
- Consider computing PFE-related probabilities as

$$\mathbb{P}(V(T) > L|\mathcal{F}(t)) = \mathbb{E}[1(V(T) > L)\mathcal{F}(t)] = \widetilde{\mathbb{E}}\left[1(V(T) > L)\frac{Z^*(T)}{Z^*(t)}|\mathcal{F}(t)\right],$$

with $Z^*(t)$ the $\widetilde{\mathbb{P}}$-into-$\mathbb{P}$ change-of-measure process.
- Error properties? Consider the expectation of the MC approximation, with $\epsilon_n = 1_n - \hat{1}_n$, $\hat{1}_n$ the AMC-based approximation to $1(V_n(T) > L)$ (recall that $\epsilon_n = \epsilon_n(N)$ as its sampling distribution depends on N).

$$\begin{aligned}\widetilde{\mathbb{E}}\left[\frac{1}{N}\sum_{n=1}^{N}\hat{1}_n\frac{Z_n^*(T)}{Z_n^*(t)}\right] &= \widetilde{\mathbb{E}}\left[\frac{1}{N}\sum_{n=1}^{N}(1_n+\epsilon_n)\frac{Z_n^*(T)}{Z_n^*(t)}\right] \\ &= \mathbb{P}(V(T) > L|\mathcal{F}(t)) + \widetilde{\mathbb{E}}\left[\epsilon(N)\frac{Z^*(T)}{Z^*(t)}\right]\end{aligned}$$

- AMC can be imprecise deep in the tails, so ϵ may be (systematically) large where Z is large if the $\mathbb{P}$- and $\widetilde{\mathbb{P}}$-measure marginals induce significantly different tails $\implies$ a possible bias for large θ? Similar for variance inflation.

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