Abandoning the Law of One Price: Economic Foundations and Mathematical Structure of Two Price Economies

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- The resulting price processes are nonlinear martingales
- Furthermore, all two price markets free of arbitrage have price processes that are nonlinearly discounted nonlinear martingales
- Maximizing the lower price yields new objective functions for portfolio selection, derivative positioning and risk management
- While, simultaneously opening up the mathematical theory for the control of nonlinearly discounted nonlinear martingales

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- There are no financial primitives in the model and finance has no economic relevance.
- In particular all can transact at prevailing prices, as a birth right.

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- As a result market clearing cannot be attained in all states.
  - Endowments can disappear.
  - Crops fail yet farmers must survive.
- The financial system exists to cover losses related to the absence of clearing.
- As a consequence the financial system (FS) defines the real economy as follows.

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- Given the necessity of loss exposures the FS must define the set of acceptable loss exposures.
- This is assumed to be some "small" cone A^* containing the nonnegative random variables.

Cones of Acceptability Characterized

- Every cone of acceptable risks is equivalently defined by a set $\mathcal{N}$ of probability measures called test scenarios and

$$X \in \mathcal{A}^* \Longleftrightarrow E^Q[X] \geq 0 \text{ all } Q \in \mathcal{N}.$$

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- Aggregate excess supplies may fail to be nonnegative but they must belong to the cone of acceptability.

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- The FS offers individual participants a “larger” cone $\mathcal{A}$ of acceptable risks defined by a smaller set of test scenarios $\mathcal{M}$.
- The FS sets prices $b(X)$, $a(X)$ for buying or selling X respectively.

The Pricing Functionals

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- In equilibrium, $\mathcal{A}$, $\mathcal{M}$ are to be solved for or determined with a view to getting aggregate excess supply into the small cone $\mathcal{A}^*$.

Technical Remarks on the Two Price Functionals

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- Unlike the classical valuation functionals of one price economies the valuation functional of a two price economy is nonlinear.
- The bid or lower price functional is concave while the upper or ask price functional is convex.
- One need only understand the details for one of these as

$$a(X) = -b(-X).$$

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- Participants of a financial economy regularly valuing human and economic activity should pay attention to the two price market value.
- The lower price functional is a concave function ideally suited to serve as an objective function for maximizing a conservative market valuation.
- We may now rewrite portfolio theory, derivative positioning, risk management and many other subjects from this new perspective.

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- As we are building a valuation functional $\mathcal{M}$ should contain a potential risk neutral measure.
- Given one risk neutral measure Q^* we ask what cones of acceptability are possible when acceptability depends on just the probability distribution function $F(x)$ of the cash flow under Q^*.

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$$\int_{-\infty}^{\infty} x d\Psi(F(x)) \geq 0.$$

- In this case

$$b(X) = \int_{-\infty}^{\infty} x d\Psi(F(x)).$$

The Set of Supporting Measures

- The set of supporting measures can be identified as the set of all measures Q satisfying

$$Q(A) \leq \Psi\left(Q^{*}(A)\right), \text{ all } A \in \mathcal{F}.$$

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- and under this choice X is acceptable just if the expectation is positive under a distribution function $G(x)$ where F is the distribution of the maximum of γ independent draws from G.

Distorted Expectation and Change of Measure

- The distorted expectation is also an expectation under a change of measure whereby

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Simulated Evaluation of Lower Price Functional

- Given a simulated outcome of cash flows x_i, $i = 1, \cdots, N$ one may evaluate the distorted expectation or lower price by ordering the outcomes in increasing order $x_{(i)}$ and evaluating

$$\sum_{i=1}^{N} x_{(i)} \left(\Psi\left(\frac{i}{N}\right) - \Psi\left(\frac{i-1}{N}\right) \right).$$

Discrete Time Roll back of Lower Price Functional

- Given a Markov environment with states x_t and transitions to states x_{t+h} in time step h with risk neutral transitional probability $Q(x_t, dx)$ and lower price functional $b(x_{t+h})$ already evaluated we define for interim cash flows $c(x_t, x_{t+h})$

$$
\begin{aligned}
b(x_t) &= E^Q\left[c\left(x_t, x_{t+h}\right) + b(x_{t+h})\right] \\
&\quad + h\int_{-\infty}^{\infty} y d\Psi\left(F(y)\right) \\
F(y) &= Q\left(\begin{array}{c} c\left(x_t, x_{t+h}\right) + b(x_{t+h}) \\ -E^Q\left[c\left(x_t, x_{t+h}\right) + b(x_{t+h})\right] \leq y \end{array}\right)
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- This pricing operator may be related to a nonlinear expectations operator yielding dynamically consistent lower price functionals.

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- and that for A known at time t that
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- is a nonlinear martingale.

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- This is shown by relating these functionals to solutions of backward stochastic difference equations as studied in Cohen and Elliott (2010, Stochastic Processes and their Applications 120, 4, 442-466).

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- For example when $X(t)$ is a pure jump Lévy process with drift term α and Lévy measure $k(y)dy$ defined for $y \neq 0$ we have that

$$\begin{aligned}\mathcal{L}(u) &= \alpha u_x(x,t) \\ &+ \int_{-\infty}^{\infty} \left(u(x+y,t) - u(x,t) - u_x(x,t)y\right) k(y)dy.\end{aligned}$$

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- In this case

$$u(x, t) = E\left[e^{-rt}\phi(X_t)|X_0 = x\right].$$

G-expectations, nonlinear expectations and nonlinear martingales

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- Peng (2006, G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Itô Type. arXiv:math/0601035v2) created $\mathcal{G}$-expectations defined as nonlinear expectations that are unique viscosity solutions to nonlinear equations of the form

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- The result is

$$u(x, t) = \mathcal{E}\left(\phi(X_t) | X_0 = x\right),$$

where $\mathcal{E}$ is a dynamically consistent nonlinear expectation operator.

G-expectations and Distortions I

- We may equivalently write for the integral in the expression for $\mathcal{L}$ the expression

$$\int_{-\infty}^{\infty} \frac{1}{y^2} \times (u(x+y,t) - u(x,t) - u_x(x,t)y) \times \left(\int_{-\infty}^{\infty} y^2 k(y)dy\right) \times g(y)dy$$

where we now write

$$g(y) = \frac{y^2 k(y)}{\int_{-\infty}^{\infty} y^2 k(y)dy}.$$

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where we now write

$$g(y) = \frac{y^2 k(y)}{\int_{-\infty}^{\infty} y^2 k(y) dy}.$$

- By construction the function $g(y)$ is positive, integrates to unity and thus is a probability density.

G-expectations and Distortions II

- Now define the variable random in y for given x, t

$$
\begin{aligned}
Y_{x,t} &= \frac{1}{y^2} \times \\
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- Also define the distribution function

$$\begin{aligned} F_{Y_{x,t}}(v) &= \int_{A(x,t,v)} g(y) dy \\ A(x,t,v) &= \left\{ \begin{array}{c} y \mid \frac{1}{y^2} \times \\ (u(x+y,t) - u(x,t) - u_x(x,t)y) \\ \times \int_{-\infty}^{\infty} y^2 k(y) dy \leq v \end{array} \right\} \end{aligned}$$

G-expectations and Distortions III

- We may then write the integral in $\mathcal{L}$ as

$$\int_{-\infty}^{\infty} v dF_{Y_{x,t}}(v).$$

Using Probability Distortions in G-expectations

- Now we introduce the distorted expectation

$$\int_{-\infty}^{\infty} v d\Psi(F_{Y_{x,t}}(v))$$

which agrees with the integral

$$-\int_{-\infty}^{0} \Psi\left(P^g\left(Y_{x,t} \leq v\right)\right) dv + \int_{0}^{\infty} \left[1 - \Psi\left(P^g\left(Y_{x,t} \leq v\right)\right)\right] dv,$$

where P^g indicates that we evaluate probability under the quadratic variation scaled density $g(y)$.

The QV operator

- Define the nonlinear operator by

$$
\begin{aligned}
\mathcal{G}_{QV}(u) &= \alpha u_x \\
&\quad - \int_{-\infty}^{0} \Psi\left(P^g\left(Y_{x,t} \leq v\right)\right) dv \\
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- This structural reweighting will be maintained on passage to measure distortions.

Measure Distortions I

- Consider now in place of an expectation an integration with respect to a positive, possibly infinite measure μ or the measure integral

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- Though the measure may be infinite, we suppose that all the tail measures are finite.
- We may then rewrite the measure integral as

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- We then define the distorted measure integral as

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\begin{aligned}
\mathfrak{m} &= -\int_{-\infty}^{0} \Gamma_+ \left(\mu\left(\left(v(y) \le x\right)\right)\right) dx \\
&\quad + \int_{0}^{\infty} \Gamma_- \left(\mu\left(\left(v(y) \ge x\right)\right)\right) dx,
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where we assume both integrals are finite.

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- **We replace measure integrals by distorted measure integrals in defining G-expectations as solutions of nonlinear partial integro-differential equations.**

Examples of Measure Distortions

- We take as models for specific measure distortions

$$\begin{aligned}\Gamma_+(x) &= x + \alpha\Psi_+\left(1 - e^{-cx}\right)\\ \Gamma_-(x) &= x - \beta\Psi_-(1 - e^{-cx})\end{aligned}$$

for any probability distortions Ψ_+, Ψ_-.

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- The lower prices are then nonlinearly discounted nonlinear martingales defined as infima of discounted conditional expectations with respect to these measures.
- Hence nonlinearly discounted nonlinear martingales are to no arbitrage in two price economies as martingales are to no arbitrage in one price economies.

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- This is operartionalized using probability and measure distortions in discrete or continuous time.
- The two price economy leads us to nonlinearly discounted nonlinear martingales and provides a new perspective to all issues in financial management.