Towards FVA Pricing

A Martingale Approach

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Context

- We have recently seen an increased interest in the " XVAs ", a series of adjustments to be added to the value of the derivative, as an attempt to fix the assumptions in derivative pricing that fail to hold during the last global financial crisis
- This presentation focuses on the Funding Value Adjustment (FVA) which has been a topic of intense discussion in the last couple of years
- The presentation introduces a formal mathematical framework that could be used to value derivative instruments in a market under funding costs and to formulate FVA
- The framework relies on traditional martingale methods adjusted to suit our purpose
- Changes include: properties of the numeraires, specification of martingale densities and the martingale measures to be used

Context

The new setting will have implications in the pricing and hedging procedures of contingent claims for financial institutions

- When selling a derivative and replicating the claim using primary instruments
- When intermediating in the market using derivatives subject to different funding costs
- When reflecting the value of the instrument in their books

Context

“ Bad artists copy, great artists steal ”
Pablo Picasso

- Bielecki and Rutkowski (2013): Market model under funding costs and required assumptions directly taken from there
- Schweizer (1992): Martingale methods applied in the present work
- Brigo et al. (2011) , Pallavecini et al. (2012), Crepey (2011), Burgard and Kjaer (2012) and others: general ideas on funding costs, XVAs, collateralization, etc.

Market Model

Assumption

We assume that:

i. The ex-dividend price processes S^i for $i = 1, .., d$ are cadlag semimartingales
ii. The dividend stream processes A^i for $i = 1, .., d$ are cadlag and of finite variation
iii. The funding account processes B^j for $j = 0, .., d$ are strictly positive, continuous and of finite variation and with $B^j_0 = 1$. Here B^0 is representing unsecured funding wether B^j for $j > 1$ represents the access to unsecured funding when holding asset j. Should be stressed that the definition of B^j is general and therefore we can define different lending and borrowing rates keeping all the proposals in the framework valid
iv. The cumulative-dividend price is given by:

$$S^{i,cld} := S^i_t + B^i_t \int_{(0,t]} (B^i_u)^{-1} dA^i_u \qquad t \in [0, T] \tag{2.1}$$

so then the discounted cumulative-dividend prices $\hat{S}^{i,cld} := (B^i_u)^{-1} S^{i,cld}$ satisfies:

$$\hat{S}^{i,cld} = \hat{S}^i_t + \int_{(0,t]} (B^i_u)^{-1} dA^i_u \qquad t \in [0, T] \tag{2.2}$$

where we denote $\hat{S}^i_t := (B^i_u)^{-1} S^i_t$

Market Model

Given the previous setting, we need a redefinition of a self-trading strategy to include funding costs. For this purpose, we make the following definition for a finite variation process A.

Definition

We say that a trading strategy (φ, A) with cash flows A is **self-financing** whenever the wealth process $V(\varphi, A)$ given by:

$$V_t(\varphi, A) := \sum_{i=1}^{d} \xi_t^i S_t^i + \sum_{j=0}^{d} \psi_t^j B_t^j \tag{2.3}$$

satisfies:

$$V_t(\varphi, A) = V_0(\varphi, A) + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i d(S_u^i + A_u^i) + \sum_{j=0}^{d} \int_{(0,t]} \psi_t^j dB_t^j + A_t \tag{2.4}$$

being $V_0(\varphi, A)$ an arbitrary real number.

We should note here that we cannot make the third term corresponding to the funding cost disappear when dividing by a unique numeraire, which is a clear break from the traditional setting.

Market Model

In order to clarify last equation, we can then redefine the terms as following:

$$G_t(\varphi, A) = \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i d(S_u^i + A_u^i) \tag{2.5}$$

as the gains (or losses) process and

$$F_t(\varphi, A) = \sum_{j=0}^{d} \int_{(0,t]} \psi_t^j dB_t^j \tag{2.6}$$

as the funding costs process; which will lead to:

$$V_t(\varphi, A) = V_0(\varphi, A) + G_t(\varphi, A) + F_t(\varphi, A) + A_t \tag{2.7}$$

Should be noted that that we call it funding costs but in reality, this could be also a benefit depending whether it will be a long or short strategy.

Reflections on Funding Costs

Proposition

The discounted wealth follows:

$$d\tilde{V}_t(\varphi, A) = \sum_{j=0}^{d} \psi_t^j d\tilde{B}_t^j + \sum_{i=1}^{d} \xi_t^j d\tilde{S}_t^{i,cld} + (B_t^0)^{-1} dA_t \tag{3.1}$$

where

$$\tilde{V}_t := (B_t^0)^{-1} V_t \tag{3.2}$$

- Proof can be found in Bielecki and Rutkowski (2013)
- Similar to standard setting but with multiple accounts
- So we cannot divide by a single savings account and eliminate the funding costs term

Reflections on Funding Costs

Before continuing the deployment of our model we find imperative to reflect on the funding costs

What are the funding costs?

- Funding costs are also present in trading strategies even in traditional modelling with only one funding account ($B^i = B$ for $i = 0, 1, ..., d$)
- Unless we can construct a perfect hedge for our contingent claim without cash and so that every single cash flow in our hedging portfolio is totally matched against the ones of our contingent claim, then there will be cash involved and hence funding costs ($\psi^0 \equiv 0$)
- The example above is more consistent with static hedging rather than dynamic hedging (although it could be theoretically possible in the later)
- However, even when $\psi^0 \neq 0$ for any self-financing hedging strategy:

$$\tilde{V}_t(\varphi, 0) = V_0(\varphi, 0) + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i d\tilde{S}_u^{i,cld} \quad (3.3)$$

This implies that in relative terms (to the savings account) we have no funding costs.

However this is not necessarily the case when $B^i \neq B^j$ for some i, j and then $F_t(\varphi, A)$ will not evolve according to the cash funding account B^0 (hence not disappearing when dividing by this numeraire).

Reflections on Funding Costs

Lemma

The following equality holds:

$$dV_t(\varphi, A) = \tilde{V}_t(\varphi, A)dB_t^0 + \sum_{i=1}^{d}(\tilde{B}_t^j)^{-1}\varsigma_t^j d\tilde{B}_t^j + \sum_{i=1}^{d}\xi_t^i B_t^i d\hat{S}_t^{i,cld} + dA_t \quad (3.4)$$

where

$$\varsigma_t^i := \psi_t^i B_t^i + \xi_t^i S_t^i \quad (3.5)$$

- This formulation hints a trick to convert this set-up into one more similar to the traditional setting; if we could eliminate the second term

Introducing Margining

To complete the analysis we introduce margining, that is collateral requirements subject to different retribution rates.

- Focus of most recent literature is on **margining**, this work contemplates **both margining and secure funding** under multiple funding accounts

When the transaction is subject to collateralization, the hedger will be either receiving or posting collateral which can be represented by the stochastic process C which will be defined by:

$$C_t = C_t \mathbb{1}_{C_t \geq 0} - C_t \mathbb{1}_{C_t < 0} = C_t^+ - C_t^- \tag{3.6}$$

- We also introduce the continuous and strictly positive processes of finite variation $B^{C,+}$ and $B^{C,-}$, representing the collateral retribution;
- and $B^{H,+}$ and $B^{H,-}$ representing the retribution of the collateral deposited in the segregation account

Introducing Margining

Definition

We say that a collateralized trading strategy is **self-financing** when its wealth process:

$$
\begin{aligned}
V_t(\varphi, A, C) \quad := \quad & \sum_{i=1}^{d} \xi_t^i S_t^i + \sum_{j=0}^{d} \psi_t^j B_t^j \\
& + \psi_t^{C,+} B_t^{C,+} + \psi_t^{C,-} B_t^{C,-} \\
& + \psi_t^{H,+} B_t^{H,+} + \psi_t^{H,-} B_t^{H,-}
\end{aligned} \tag{3.7}
$$

satisfies:

$$
\begin{aligned}
V_t(\varphi, A) \quad = \quad & V_0(\varphi, A) + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i d(S_u^i + A_u^i) + \sum_{j=0}^{d} \int_{(0,t]} \psi_t^j dB_t^j + A_t \\
& + \int_{(0,t]} \psi_u^{C,+} dB_u^{C,+} + \int_{(0,t]} \psi_u^{C,-} dB_u^{C,-} \\
& + \int_{(0,t]} \psi_u^{H,+} dB_u^{H,+} + \int_{(0,t]} \psi_u^{H,-} dB_u^{H,-}
\end{aligned} \tag{3.8}
$$

Introducing Margining

Then defining the margining costs as:

$$F_t^M(\varphi, A, C) \quad := \quad \int_{(0,t]} \psi_u^{C,+} dB_u^{C,+} + \int_{(0,t]} \psi_u^{C,-} dB_u^{C,-} \\ + \int_{(0,t]} \psi_u^{H,+} dB_u^{H,+} + \int_{(0,t]} \psi_u^{H,-} dB_u^{H,-} \tag{3.9}$$

We have that for any any self-financing trading strategy:

$$V_t(\varphi, A) = V_0(\varphi, A) + G_t(\varphi, A) + F_t(\varphi, A) + F_t^M(\varphi, A) + A_t \tag{3.10}$$

Note that we have added here an extra term F^m corresponding to the Profit and Loss generated by the margining requirements.

Introducing Margining

Assumption

We assume in the present work that cash collateral will be used with full segregation. This is whenever the hedger receives collateral $\psi^{C,+}$ which will be remunerated at $B^{C,+}$, this will be held in a segregated account and therefore the hedger cannot use it to invest in risky assets but rather deposited all into $B^{H,+}$ which will be reflected in the position $\psi^{H,+}$. We can make the inverse argument for the hedger posting collateral $\psi^{C,-}$ at $B^{C,-}$ which will be funded by $\psi^{H,-}$ at $B^{H,-}$; and hence we get the following:

$$\psi^{C,+}B^{C,+} + \psi^{H,+}B^{H,+} = 0 \tag{3.11}$$

$$\psi^{C,-}B^{C,-} + \psi^{H,-}B^{H,-} = 0 \tag{3.12}$$

with

$$\psi_t^{C,+} = \left(B_t^{C,+}\right)^{-1} C_t^+ \qquad \psi_t^{C,-} = \left(B_t^{C,-}\right)^{-1} C_t^- \tag{3.13}$$

$$\psi_t^{H,+} = \left(B_t^{H,+}\right)^{-1} C_t^+ \qquad \psi_t^{H,-} = \left(B_t^{H,-}\right)^{-1} C_t^- \tag{3.14}$$

This assumption also implies $F^M(\varphi, A) = F^M(A)$ that is the funding costs do not depend on the trading strategy (hence neither on the asset positions held) This makes us capable to join together F^M and A, making it a unique external cash flow to be replicated.

Introducing Margining

Assumption

The following condition holds:

$$\psi_t^i B_t^i = -\xi_t^i S_t^i \tag{3.15}$$

implying $\varsigma = 0$. We will refer to this as the " Repo " condition

This assumption will not hold in practice but any secured funding will be subject to a haircut depending on the asset type. However, it is relatively easy to make an adjustment to reflect the proportion of the asset position held available for secured funding as long as the haircut will be independent of the asset position.

No-Arbitrage Price

Theorem

If exists $\tilde{\mathbb{P}}$ such that $S^{i,cld}$ for $i = 1, ..., d$ are $(\tilde{\mathbb{P}}\text{-}\mathbb{G})$-local martingales, then the model is arbitrage free

Proof

$$V_t^{cld}(\varphi, A) := V_t(\varphi, A) - B_t^0 \int_{(0,t]} (B_u^0)^{-1} dA_u \tag{3.16}$$

$$\tilde{V}_t(\varphi, A) := (B_u^0)^{-1} V_t(\varphi, A) \tag{3.17}$$

$$\tilde{V}_t^{cld}(\varphi, A) = \tilde{V}_0^{cld}(\varphi, A) + \sum_{j=0}^{d} \int_{(0,t]} (B_u^0)^{-1} B_t^i \xi_t^i d\hat{S}_t^{i,cld} \tag{3.18}$$

Note that the integrator for the second term is a cadlag function and the integrand a $(\tilde{\mathbb{P}}\text{-}\mathbb{G})$-local martingale so then as $\tilde{V}_t^{cld}(\varphi, A)$ is admissible and hence bounded from below, then $\tilde{V}_t^{cld}(\varphi, A)$ is a supermartingale

No-Arbitrage Price

Definition

We say that a trading strategy **replicates** a contract with cash flows D whenever $A_t = D_t$ for every $t \in [0, T]$ and $V_T(\varphi, A) = 0$, then $P(\varphi) = V(\varphi, A)$ denotes the ex-dividend price for the claim.

Here D and A are equivalent, however we want to highlight the fact that A_t are the cash flows of the strategy while D_t are those of the contingent claim to be hedged.

Theorem

Assume D can be replicated by a trading strategy (φ, A) , then the ex-dividend price is:

$$P_t(\varphi) = -B_t^0 \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_{(t,T]} (B_u^0)^{-1} dD^M | \mathcal{G}_t \right] \tag{3.19}$$

where $D^M := F^M + D$ denotes the associated cash flows inclusive of margining costs as by full segregation assumption the margining costs are independent of the trading strategy.

No-Arbitrage Price

Proof:
We have that:

$$\tilde{V}_T(\varphi, A) - \tilde{V}_t(\varphi, A) = \sum_{i=1}^{d} \int_{(t,T]} (B_u^0)^{-1} \xi_t^i B_t^i d\hat{S}_t^{i,cld} + \int_{(t,T]} (B_u^0)^{-1} dD_t^M \quad (3.20)$$

Taking conditional $\tilde{\mathbb{P}}$ expectation on both sides with respect to $\mathcal{G}$ gives us:

$$\tilde{V}_t(\varphi, A) = -\mathbb{E}_{\tilde{\mathbb{P}}} \left[\sum_{i=1}^{d} \int_{(t,T]} (B_u^0)^{-1} \xi_t^i B_t^i d\hat{S}_t^{i,cld} + \int_{(t,T]} (B_u^0)^{-1} dD_t^M | \mathcal{G}_t \right] \quad (3.21)$$

As each $\hat{S}_t^{i,cld}$ are ($\tilde{\mathbb{P}}$-$\tilde{\mathbb{G}}$) local martingales then the first term in the expectation disappears.

No-Arbitrage Price

Before jumping into the martingale approach it will be important to reflect on the steps taken so far:

- We have " Magically " got to an expression for the trading strategy similar to the one in the traditional setting, owing this magic to the Repo and Full-Segregation assumptions
- This allowed us to express the price of a contingent claim as an expectation type formula
- However, without much notice, this new setting implies important changes to the traditional setting as we shall see.

Martingale Densities

To facilitate notation, as its abuse is almost unavoidable in this setting, we make $Z = Z^{\tilde{\mathbb{P}}}$.

Definition

A $(\mathbb{P},\mathbb{G})$-local martingale Z with $Z_0 = 1$ is a **martingale density** for $\tilde{S}^{i,cld}$ if the process $Z\tilde{S}^{i,cld}$ is also a $(\mathbb{P},\mathbb{G})$-local martingale. Z is a strict martingale density if in addition it is strictly positive (and hence a supermartingale).

Remark

If Z is a strict martingale density and $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$; then $\tilde{\mathbb{P}}$ is a probability measure (i.e. $\mathbb{P}(\Omega) = 1$) if and only if Z is a uniformly integrable martingale which is only true if $\mathbb{E}_{\mathbb{P}}\left[Z_T\right] = 1$

Martingale densities are slightly more general processes than the equivalent martingale measures, therefore, we will keep the mathematical development in terms of the martingale densities as it is easier to restrict ourselves at a later stage to the case of equivalent martingale measures.

Martingale Densities

Assumption

We assume that the ex-dividend price processes S^i are all special semimartingales under $\mathbb{P}$ with canonical decomposition:

$$S_t^i = S_0^i + M^i + A^i \tag{4.1}$$

where M^i are $(\mathbb{P}\text{-}\mathbb{G})$ local martingales and A^i process of finite variation with $M_0 = A_0 = 0$

Proposition

Under this assumption the discounted cumulative dividend prices $\hat{S}^{i,cld}$ are also special semimartinagales under $\mathbb{P}$.

We can then reexpress the discounted cumulative dividend price process also as a special martingale :

$$\hat{S}_t^{i,cld} = S_0^i + \hat{M}_t^{i,cld} + \hat{A}_t^{i,cld} \tag{4.2}$$

where the martingale is defined by:

$$\hat{M}_t^{i,cld} := \int_{(0,t]} (B_t^i)^{-1} dM_t^i \tag{4.3}$$

and the finite variation process as:

$$\hat{A}_t^{i,cld} := \int_{(0,t]} (B_t^i)^{-1} dA_t^i - \int_{(0,t]} (B_t^i)^{-2} S_{t-}^i dB_t^i + \int_{(0,t]} (B_t^i)^{-1} dD_t^i \tag{4.4}$$

Martingale Densities

Definition

Let Z be a martingale density, then we define the **state price density** process ϱ as the d-dimensional process $\varrho = (\varrho^1, ..., \varrho^d)$ where:

$$\varrho^i = (B^i)^{-1} Z \qquad i = 1, ..., d \tag{4.5}$$

Remark

We find relevant to mention under funding costs, the state density process cannot longer be represented by a one dimensional process as the funding account is not unique and therefore should take the form of a d-dimensional process.

Martingale Densities

The next two propositions are required to prove the following Lemma.

Proposition

Let Z be a strict martingale density for $\hat{S}^{cld}$, ϱ a d-dimensional process representing the state price density for $\tilde{S}^{cld}$ and $\tilde{M}^{cld}\varrho$ a d-dimensional special semimartingale, then

$$\int_{(0,t]} -B^i_u (Z_{t-})^{-1} (S^i_{u-})^{-1} d\left\langle S^i, \varrho^i \right\rangle^{\mathbb{P}}_u = \int_{(0,t]} (S^i_{u-})^{-1} dA^i_u - \int_{(0,t]} (B^i_u)^{-1} dB^i_u + \int_{(0,t]} (S^i_{u-})^{-1} dD^i_u \quad (4.6)$$

Proposition

Suppose Z is a strict martingale density for $\hat{S}^{cld}$ and Z and $\hat{M}^{i,cld}$ are locally square integrable, then $\hat{A}^{i,cld}$ is absolutely continuous with respect to $\left\langle \hat{M}^{i,cld} \right\rangle^{\mathbb{P}}$ that is:

$$\hat{A}^{i,cld}_t = \int_{(0,t]} d\hat{A}^{i,cld}_u = \int_{(0,t]} \hat{\alpha}^{i,cld}_u d\left\langle \hat{M}^{i,cld} \right\rangle^{\mathbb{P}}_u \quad (4.7)$$

for some process $\hat{\alpha}^{i,cld}$ integrable with respect to a local martingale.

Martingale Densities

Lemma

Suppose $\hat{S}^{cld}$ *is a d-dimensional special semimartingale with the finite variation processes of the canonical decomposition for each* $\hat{S}^{i,cld}$ *represented by* $\hat{A}^{i,cld} = \int \hat{\alpha}^{i,cld} d\left\langle \hat{M}^{i,cld} \right\rangle^{\mathbb{P}}$ *then* Z *is a martingale density*

if and only if exists a $(\mathbb{P},\mathbb{G})$*-local martingale* N^i $(N_0^i = 0)$ *orthogonal to* $\hat{M}^{i,cld}$ $(N^i \hat{M}^{i,cld}$ *is a* $(\mathbb{P},\mathbb{G})$*-local martingale) such that:*

$$Z_t = 1 - \int_{(0,t]} Z_{u-} \hat{\alpha}_u^{i,cld} d\hat{M}_u^{i,cld} + N_t^i \qquad (4.8)$$

Martingale Densities

Now, that we have proved the existence of the martingale measure given certain conditions, we want to find the expression for this martingale measure.

In particular we are interested in a process $\hat{\lambda}$ representing the martingale density so that:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}\left(\int^{\cdot} \sum_{i=1}^{d} \hat{\lambda}_u d\hat{M}_u^{i,cld}\right)_T \tag{4.9}$$

where $\mathcal{E}$ represents the stochastic exponential as usual.

- Given we are not specifying the uniqueness of the process $\hat{\lambda}$ (and therefore also the martingale measure is not necessarily unique), we allow for the setting to be still valid in an incomplete market framework

Martingale Densities

Theorem

Let $\hat{\lambda} = (\hat{\lambda}^1, ..., \hat{\lambda}^d)$ *and* $\gamma = (\gamma^1, ..., \gamma^{1\,d})$ *be d-dimensional processes and* $\sigma \in \mathbb{R}^d \otimes \mathbb{R}^d$.

Define:

$$\beta_t := \sum_{i=1}^{d} \left\langle \hat{M}^{i,cld} \right\rangle_t^{\mathbb{P}} \tag{4.10}$$

$$\sigma_t^{ij} := \frac{d\left\langle \hat{M}^{i,cld}, \hat{M}^{i,cld} \right\rangle_t^{\mathbb{P}}}{d\beta_t} \tag{4.11}$$

$$\gamma_t^i := \hat{\alpha}_t^{i,cld} \sigma_t^{ii} \tag{4.12}$$

Suppose that:

i.
$$\sigma_t \quad \text{is invertible} \tag{4.13}$$

ii.
$$\hat{\lambda} := \sigma_t^{-1}\gamma_t \quad \text{is a suitable integrator for } \hat{M}^{i,cld} \tag{4.14}$$

Then Z satisfies:

$$Z_t = 1 - \int_{(0,t]} Z_{u-}\hat{\lambda}_u \cdot d\hat{M}_u^{cld} + R_t \tag{4.15}$$

for a process R_t *orthogonal to* $\tilde{M}^{cld}$

Martingale Densities

Corollary

If we define:

$$N^i := R_t + \int_{(0,t]} Z_{u-}\hat{\alpha}_u^{i,cld}\, d\hat{M}_u^{i,cld} - \int_{(0,t]} Z_{u-}\hat{\lambda}_u \cdot d\hat{M}_u^{cld} \tag{4.16}$$

we have that N^i and $\hat{M}^{i,cld}$ are orthogonal.

This is obvious combining last theorem with the previous lemma.

Martingale Densities

We can make the expression more prescriptive and define the martingale density in terms of the ex-dividend price

$$\hat{\alpha}_u^{i,cld} = (B_t^i)\left(d\left\langle M^i\right\rangle_u^{\mathbb{P}}\right)^{-1}\left(dA_t^i - (B_t^i)^{-1}S_{t-}^i dB_t^i + dD_t^i\right) \tag{4.17}$$

$$\beta_t^i = \sum_{i=1}^{d}\int_{(0,t]}(B_u^i)^{-2}d\left\langle M^i\right\rangle_u^{\mathbb{P}} \tag{4.18}$$

$$\sigma_t^{ij} = \left(\sum_{i=1}^{d}(B_t^i)^{-2}d\left\langle M^i\right\rangle_t^{\mathbb{P}}\right)^{-1}(B_t^i)^{-1}(B_t^j)^{-1}d\left\langle M^i, M^j\right\rangle_t^{\mathbb{P}} \tag{4.19}$$

$$\gamma_t^i = \left(dA_t^i - (B_t^i)^{-1}S_{t-}^i dB_t^i + dD_t^i\right)\left((B_t^i)\sum_{i=1}^{d}(B_t^i)^{-2}d\left\langle M^i\right\rangle_t^{\mathbb{P}}\right)^{-1} \tag{4.20}$$

Example: An Ito process

Assumption

The ex-dividend price follows:

$$dS^i = a^i(S,t)dt + b^i(S,t)\cdot dW \tag{4.21}$$

where $W = (W^1, ..., W^d)$ is a d-dimensional Brownian motion.
If B^i and D^i follow

$$dB_t^i = B_t^i r_t^i dt \tag{4.22}$$

$$dD_t^i = S_t^i \delta_t^i dt \tag{4.23}$$

Then we can write:

$$\sigma_t^{ij} = \left(\sum_{i=1}^{d} (B_t^i)^{-2} b^i(s,t)^2\right)^{-1} (B_t^i)^{-1}(B_t^j)^{-1} b^i(s,t) b^j(s,t) \tag{4.24}$$

$$\gamma_t^i \quad = \quad (a^i(S,t) + S_t^i\delta_t^i - S_t^i r^i)\left((B_t^i)\sum_{i=1}^{d}(B_t^j)^{-2}(b^i(s,t))^2\right)^{-1} \tag{4.25}$$

As a checkpoint against the well known traditional setting we can make $d = 1$:

$$\hat{\lambda}_t = \hat{\lambda}_t^1 = (B_t^1)\left(a^1(S,t) + S_t^1\delta_t^1 - S_t^1 r_t^1\right)(b^1(s,t))^{-2} \tag{4.26}$$

$$\hat{\lambda}_u \cdot d\hat{M}_u^{cld} = \left(\frac{a^1(S,t)}{S_t^1} + \delta_t^1 - r_t^1\right)\left(\frac{b^1(S,t)}{S_t^1}\right)^{-1}$$

FVA definition

Now that we have built a robust mathematical framework, we are able to attempt an FVA definition:

Definition

We can define the Funding Value Adjustment (FVA) as the difference between the price of a contingent claim not subject to any funding costs (Π) and the price of the same claim when considering those funding costs (P)

$$FVA = \Pi - P \tag{5.1}$$

The issue in the present context is that two possible definitions then arise.

FVA definition

If we consider no funding costs as zeroing the collateral requirements (this is customary in current research).
This is equivalent to assume $C \equiv 0$, so then $F^M = 0$ or equivalently

$$\int_{(0,t]} \psi_u^{C,+} dB_u^{C,+} + \int_{(0,t]} \psi_u^{C,-} d + \int_{(0,t]} \psi_u^{H,+} dB_u^{H,+} + \int_{(0,t]} \psi_u^{H,-} dB_u^{H,-} = 0 \qquad (5.2)$$

Then, assuming that the " Repo " assumption still holds we get:

$$\Pi^1 = -B_t^0 \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_{(t,T]} (B_u^0)^{-1} dD | \mathcal{G}_t \right] \qquad (5.3)$$

which makes

$$FVA^1 = B_t^0 \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_{(t,T]} (B_u^0)^{-1} dF^M | \mathcal{G}_t \right] \qquad (5.4)$$

- We should note here that although the approach looks quite familiar, the martingale measure $\tilde{\mathbb{P}}$ is the one that makes $\hat{S}^{i,cld}$ martingales
- This approach is consistent with some academic research but not entirely consistent with market practice, as the FVA^1 it would only applicable when the price has been calculated as Π^1

FVA definition

The additional assumption $B^i \equiv B^0$ is required to arrive to a definition more consistent with common practice of adding FVA charge on top of the "*traditional*" valuation formulas with no funding costs.
When the later holds we have that:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{d\mathbb{P}^*}{d\mathbb{P}} \tag{5.5}$$

where $\mathbb{P}^*$ is the traditional equivalent martingale measure for any asset discounted by the (univariate) numeraire B^0. This gives us:

$$\Pi^2 = -B_t^0 \mathbb{E}_{\mathbb{P}*}\left[\int_{(t,T]} (B_u^0)^{-1} dD | \mathcal{G}_t\right] \tag{5.6}$$

which makes

$$FVA^2 = B_t^0 \mathbb{E}_{\tilde{\mathbb{P}}}\left[\int_{(t,T]} (B_u^0)^{-1} dD^M | \mathcal{G}_t\right] - B_t^0 \mathbb{E}_{\mathbb{P}^*}\left[\int_{(t,T]} (B_u^0)^{-1} dD | \mathcal{G}_t\right] \tag{5.7}$$

Conclusions

- The two FVA formulations will provide different answers
- When we start defining FVA with an expectation or assume a particular martingale measure this difference is hard to observe
- There is no clear alignment between theory and practice
- The implications are relevant from a pricing, hedging and risk management perspective

Thank you!